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AXISYMMETRIC TORSION OF AN ELASTIC SPACE WITH A THIN ELASTIC INCLUSION*

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The state of stress and strain of an elastic isotropic space with a thin foreign disc-like inclusion whose edge has the shape of a small-aperture angle is investigated by the method of combined asymptotic expansions (CAE) /1-3/. The elastic system is in a state of axisymmetric torsion. The principal terms are obtained for the asymptotic expansions of the solution of the problem in a small parameter that characterizes the relative thickness of the inhomogeneity.

1. Formulation of the problem and method of solution. In an elastic isotropic space let there be a thin disc-like inclusion that occupies the domain

$$W_\varepsilon = \{(r; z): r \in [0; 1]; \varepsilon f_2(r) \leqslant z \leqslant \varepsilon f_1(r)\}; f_2(r) \leqslant f_1(r)$$
$$|f_i(r)| < C, \ |f_i'(r)| < C, \ C = \text{const}, \ 0 \leqslant r \leqslant 1, \ i = 1, 2$$

(r, θ, z are dimensionless cylindrical coordinates, $f_i(r)$ are sufficiently smooth functions for $0 \leqslant r < 1$, and ε is a small positive dimensionless parameter). The edge of the inclusion has the shape of a small-aperture angle so that

$$\begin{aligned}&f_i(r) = b_i(1-r) + o(1-r), \ r \to 1-0, \ i = 1, 2; \ b_i = \text{const}\\&\varepsilon g(r) = \varepsilon[f_1(r) - f_2(r)] = \varepsilon b(1-r) + o(1-r), \quad r \to 1-0\\&b = b_1 - b_2\end{aligned} \tag{1.1}$$

($\varepsilon g(r)$ is the variable thickness of the inclusion).

Under axisymmetric torsion, the displacement vector component different from zero in an elastic body, the tangential displacement, satisfies the equation /4/

$$\begin{aligned}&\frac{\partial^2 u_\theta}{\partial z^2} + Lu_\theta = 0, \quad L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\\&u_\theta(r, z) = \begin{cases} u_\theta^i(r, z) + u_*(r, z), & (r, z) \in \Omega_\varepsilon, \quad (-1)^i z \leqslant 0\\ u_\theta^0(r, z), & (r, z) \in W_\varepsilon\end{cases}\\&u_\theta^i(r, z) \to 0, \quad \sqrt{r^2 + z^2} \to \infty; \quad i = 1, 2\end{aligned} \tag{1.2}$$

Here Ω_ε is the domain of the body outside the inclusion, and $u_*(r, z)$ are unknown tangential displacements in the body when the elastic properties of the host and inclusion materials are identical.

The matching conditions on the material interfacial boundary are written as follows (it is assumed that the displacements and stresses are continuous during passage across the interfacial surface):

$$u_\theta^i + u_* = u_\theta^0, \quad T_n(u_\theta^i + u_* - \gamma u_\theta^0) = 0, \quad z = \varepsilon f_i(r) \tag{1.3}$$

$$0 \leqslant r < 1, \ i = 1, 2; \ \gamma = \mu_0/\mu$$

$$u_\theta^1 = u_\theta^2, \quad \frac{\partial}{\partial z}u_\theta^1 = \frac{\partial}{\partial z}u_\theta^2, \quad z = 0, \quad r > 1 \tag{1.4}$$

$$T_n(u) = \cos(r, n)\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) + \cos(z, n)\frac{\partial u}{\partial z} \tag{1.5}$$

$$\cos(z, n) = (-1)^{i+1} + o(1),$$
$$\cos(r, n) = (-1)^{i+1}\varepsilon\frac{\partial}{\partial r}f_i(r) + o(\varepsilon), \quad \varepsilon \to 0$$

where μ_0 and μ are the shear moduli of the inclusion and matrix materials, respectively, and $\cos(r, n)$ and $\cos(z, n)$ are the cosines of the angles between the external normal and to the inclusion contour and the r and z axes.

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Furthermore, the CAE method /1-3/ is used to seek the principal terms of the state of stress and strain of a composite which are, as a rule, of greatest interest in applications. It turns out that the solution for different values of the parameter $\gamma = \mu_0/\mu$ $(0 \leqslant \gamma < \infty)$ cannot successfully be described in a unique manner. Consequently, it is proposed to conduct a separate investigation of the problem here for three ranges of variation of the parameter γ:

$$1)\ \sqrt{\varepsilon} \leqslant \gamma \leqslant 1/\sqrt{\varepsilon}, \quad 2)\ 0 \leqslant \gamma \leqslant \sqrt{\varepsilon}, \quad 3)\ 1/\sqrt{\varepsilon} \leqslant \gamma < \infty \tag{1.6}$$

Th external and internal asymptotic expansions are constructed in each range. It will be shown that domains of values of γ exist in which the solutions for adjacent ranges overlap with a certain degree of accuracy. We note that the partition (1.6) is provisional; it can be realized by another method.

2. Construction of the external asymptotic expansion.

The external formal expansion of the solution of the problem formulated describes the state of stress and strain in the whole composite with the exception of a small neighbourhood of the edge of the inclusion whose dimensions, are indicated below. We represent this expansion in the form

$$\begin{aligned} &u_\theta{}^i(r, z) = u_{j0}{}^i(r, z) + \varepsilon u_{j1}{}^i(r, z) + \ldots, \quad \varepsilon \to 0 \\ &u_\theta{}^0(r, z) = u_{j0}{}^0(r, z_*) + \varepsilon u_{j1}{}^0(r, z_*) + \ldots, \quad \varepsilon \to 0, \quad z = \varepsilon z_* \end{aligned} \tag{2.1}$$

($j = 1, 2, 3$ is a subscript indicating the range of the parameter γ from (1.6)).

We will use the variables r and $z_* = z/\varepsilon$ to describe the solution in the domain of the inclusion. Then (1.2) takes the form

$$\frac{\partial^2}{\partial z_*^2} u_\theta{}^0 + \varepsilon^2 L u_\theta{}^0 = 0, \quad (r, z_*) \in W_\varepsilon \tag{2.2}$$

Substituting (2.1) into (1.2) and (2.2) and equating the expressions for identical powers of ε, we obtain for the first two approximations

$$\frac{\partial^2}{\partial z^2} u_{jk}{}^i + L u_{jk}{}^i = 0; \quad u_{jk}{}^i \to 0, \quad \sqrt{r^2 + z^2} \to \infty; \quad i = 1, 2 \tag{2.3}$$

$$u_{jk}{}^1 = u_{jk}{}^2, \quad \frac{\partial}{\partial z} u_{jk}{}^1 = \frac{\partial}{\partial z} u_{jk}{}^2, \quad z = 0, \quad r > 1; \quad j = 1, 2, 3 \tag{2.4}$$

$$\frac{\partial^2}{\partial z_*^2} u_{j0}{}^0 = \frac{\partial^2}{\partial z_*^2} u_{j1}{}^0 = \frac{\partial^2}{\partial z_*^2} u_{j2}{}^0 + L u_{j0}{}^0 = 0, \quad (r, z_*) \in W_\varepsilon \tag{2.5}$$

From (2.5) we have ($A_{jk}(r)$ and $B_{jk}(r)$ are unknown functions)

$$\begin{aligned} &u_{jk}{}^0 = z_* A_{jk}(r) + B_{jk}(r) \ (k = 0, 1), \quad u_{j2}{}^0 = z_* A_{j2}(r) + B_{j2}(r) - \\ &\quad \frac{1}{6} z_*^2 L\,[z_* A_{j0}(r) + 3B_{j0}(r)], \quad (r, z_*) \in W_\varepsilon, \quad j = 1, 2, 3 \end{aligned} \tag{2.6}$$

Our further description will be for each range of variation of the parameter from (1.6) separately.

Let us examine the range *1*. We will first assume that

$$\gamma = O(1),\ 1/\gamma = O(1),\ \varepsilon \to 0 \tag{2.7}$$

Writing the conjugate conditions (1.3) for series (2.1) and equating the coefficients of identical powers of ε, we obtain the following conjugate conditions when (1.5), (2.6) and (2.7) are taken into account /5/:

$$u_{10}{}^1 = u_{10}{}^2, \quad \frac{\partial}{\partial z} u_{10}{}^1 = \frac{\partial}{\partial z} u_{10}{}^2, \quad z = 0, \quad r < 1 \tag{2.8}$$

$$u_{11}{}^1 - u_{11}{}^2 = 2\psi_2(r), \quad \frac{\partial}{\partial z} u_{11}{}^1 - \frac{\partial}{\partial z} u_{11}{}^2 = 2\psi_1(r), \quad 0 \leqslant r < 1, \quad z = 0 \tag{2.9}$$

$$\psi_1(r) = (1 - \gamma) \frac{\partial}{2r^2 \partial r} \left[r^3 g(r) \frac{\partial}{\partial r} \left(\frac{1}{r} u_*(r, 0) \right) \right]$$

$$\psi_2(r) = \frac{1 - \gamma}{2\gamma} g(r) \frac{\partial}{\partial z} u_*(r, 0)$$

$$\begin{aligned} &A_{10}(r) = 0, \quad B_{10}(r) = u_*(r, 0), \quad A_{11}(r) = \frac{1}{\gamma} \frac{\partial}{\partial z} u_*(r, 0) \\ &B_{11}(r) = u_{11}{}^1(r, 0) + f_1(r) \left[\frac{\partial}{\partial z} u_*(r, 0) - A_{11}(r) \right] \end{aligned} \tag{2.10}$$

The relationships (2.3), (2.4), (2.6), (2.8)-(2.10) completely define the principal term of the external expansion (2.1) obtained under condition (2.7). It can be shown by using the results from /2, 3, 6/ that the asymptotic expansions of the tangential displacements have the form

$$u_\theta^i(r, z) = \varepsilon u_{11}^i(r, z) + O(\varepsilon^2\gamma^2) + O(\varepsilon^2/\gamma^2), \ \varepsilon \to 0, \ i = 1, 2 \tag{2.11}$$
$$u_\theta^0(r, z) = u_{10}^0(r, z_*) + \varepsilon u_{11}^0(r, z_*) + O(\varepsilon^2\gamma^2) + O(\varepsilon^2/\gamma^2)$$
$$\varepsilon \to 0$$

It is seen from the estimates presented, as well as from (2.9) and (2.10), that the representations (2.11) hold for all values of γ in the range *1* from (1.6) and cease to be suitable for $\gamma < C\varepsilon$ or $\gamma > C/\varepsilon$, $C = \text{const}$, $\varepsilon \to 0$. Consequently, ranges of variation *2* and *3* of the parameter γ from (1.6) are additionally examined, in which the shear modulus of the inclusion differs significantly from the shear modulus of the medium.

The conjugate conditions for determining the principal terms of the external asymptotic expansions in cases when the elastic properties of the inclusion and matrix materials differ substantially are found exactly like relationships (2.8)-(2.10), except that it is assumed in place of (2.7) that $\gamma = \mu_0/\mu \to 0$, $\varepsilon \to 0$, in range *2* and $\gamma \to \infty$, $\varepsilon \to 0$ in range *3*. The conjugate conditions mentioned have the form

$$2\gamma_2(u_{20}^1 - u_{20}^2) = g(r)\frac{\partial}{\partial z}(u_{20}^1 + u_{20}^2 + 2u_*) \tag{2.12}$$
$$\frac{\partial}{\partial z}u_{20}^1 = \frac{\partial}{\partial z}u_{20}^2, \quad 0 \leqslant r < 1, \quad z = 0, \quad \gamma_2 = \frac{\gamma}{\varepsilon}$$
$$u_{20}^0(r, z_*) = g^{-1}(r)[u_{20}^1(r, 0) - u_{20}^2(r, 0)][z_* - f_1(r)] + u_{20}^1(r, 0) + u_*(r, 0), \quad (r, z_*) \in W_\varepsilon \tag{2.13}$$
$$\frac{\partial}{\partial z}u_{30}^1(r, 0) - \frac{\partial}{\partial z}u_{30}^2(r, 0) = \tag{2.14}$$
$$-\gamma_3\frac{\partial}{2r^2\partial r} \times \left[g(r)r^3\frac{\partial}{\partial r}(u_{30}^1 + u_{30}^2 + 2u_*)\right]$$
$$u_{30}^1 = u_{30}^2, \quad 0 \leqslant r < 1, \quad z = 0, \quad \gamma_3 = \varepsilon\gamma$$
$$u_{30}^0(r, z_*) = u_{30}^1(r, 0) + u_*(r, 0), \quad (r, z_*) \in W_\varepsilon \tag{2.15}$$

Relationships (2.3), (2.4), (2.12)-(2.15) completely define the principal terms u_{20}^i and u_{30}^i of the external expansions in ranges *2* and *3*.

An asymptotic analysis of spatial elasticity theory problems analogous to those under consideration was performed, for example, in /6-8/, for bodies containing more compliant or stiffer thin elastic inclusions than the matrix medium. Relationships (2.12) and (2.14) agree with the conditions obtained by another method /6, 7/ if they are written in a form corresponding to the case of the torsion of an isotropic elastic body. We mention that the abovementioned relationships are equivalent to the analogous conditions used in /9-11/. Different models of thin inclusions in an elastic medium are presented in /12-14/ for the plane problem of the theory of elasticity.

Following /2, 3, 6/, it can be shown that outside a certain small neighbourhood of the end of the inclusion the following estimates hold:

$$u_\theta^i(r, z) = u_{20}^i(r, z) + O(\varepsilon), \ \varepsilon \to 0, \ \gamma \to 0 \tag{2.16}$$
$$u_\theta^i(r, z) = u_{30}^i(r, z) + O(\varepsilon), \ \varepsilon \to 0, \ 1/\gamma \to 0$$
$$(u_{j0}^i(r, z) = O(1/\varkappa_j), \ 1/\varkappa_j \to 0, \ \varepsilon \to 0, \ j = 2, 3; \ \varkappa_2 = 2\gamma_2,$$
$$\varkappa_3 = 2/\gamma_3)$$

where the parameters γ_j are defined according to relationships (2.12) and (2.14). It follows from the estimates presented that the representations (2.16) lose the asymptotic nature when $\gamma = O(1)$, $1/\gamma = O(1)$, $\varepsilon \to 0$. On the basis of (2.11) and (2.16) we see the existence of domains of variation of the parameter γ in which the solutions obtained in adjacent ranges of (2.16) overlap. It is also seen that the partition into ranges can be performed by other methods also.

The external representations obtained do not describe the solutions of the problem under consideration in a small neighbourhood of the end of the inclusion. By analogy with /1/ it can be shown that the size of this domain is determined by the order $\exp(-C/\varepsilon)$, where $C = \text{const}$, $\varepsilon \to 0$. To refine the state of stress and strain in the neighbourhood mentioned somewhat later, we construct the internal asymptotic expansion.

3. Solution of external problems.

We now consider the solution of external problems, i.e., the determination of the functions u_{j0}^i, u_{j1}^j ($j = 1, 2, 3$; $i = 1, 2$).

By using the Hankel transform /15/, we determine from relationships (2.3), (2.4), (2.8) and (2.9)

$$u_{10}{}^i(r,z)\equiv 0,\quad r\geqslant 0,\quad |z|\geqslant 0,\quad i=1,2;\quad u_{11}{}^1(r,z)=\sum_{n=1}^{2}\varkappa_n{}^i\times \tag{3.1}$$
$$\int_0^1 \psi_n(t)K_n(t,r,z)\,dt;\qquad K_n(t,r,z)=\int_0^\infty \eta^{n-1}J_1(\eta t)J_1(\eta r)e^{-\eta|z|}\,d\eta,\qquad n=1,2$$

($\psi_i(t)$ $(i=1,\ 2)$ are functions in the relationships (2.9) $\varkappa_1{}^i=-1,\ \varkappa_2{}^i=(-1)^{i+1}$).

It follows here from the results in /7, 16/ that near the edge of the inclusion

$$u_* + u_1 \backsim D_0 + \varepsilon D_{10} + \rho\,(C_1\sin\varphi + D_1\cos\varphi) + \varepsilon\psi\,(\rho,\ \varphi), \tag{3.2}$$
$$\rho\to 0$$
$$0\leqslant\varphi\leqslant 2\pi,\quad \psi=O(\rho\ln\rho);\quad D_{10}=\int_0^1\psi_1(t)K_2(t,1,0)\,dt$$
$$u_1(r,\ z)=\varepsilon u_{11}{}^1(r,\ z),\ z\geqslant 0;\ u_1(r,\ z)=\varepsilon u_{11}{}^2(r,\ z),\ z\leqslant 0$$
$$1-r=\rho\cos\varphi,\ z=\rho\sin\varphi$$
$$u_*(\rho,\ \varphi)\backsim D_0+\rho\,(C_1\sin\varphi+D_1\cos\varphi)+\ \ldots,\ \rho\to 0 \tag{3.3}$$

($\rho,\ \varphi$ are polar coordinates and $D_0,\ C_1,\ D_1$ are known coefficients).

The functions $u_{20}{}^i(r,\ z),\ u_{30}{}^i(r,\ z)$ are determined from the relationships (2.3), (2.4), (2.12) and (2.14). Analogous problems were examined in /10, 11, 17, 18/, for instance. Using the Hankel transform, we write

$$u_{j0}{}^i(r,z)=(-1)^k\int_0^1 t\varphi_j(t)K_n(t,r,z)\,dt,\quad k=(i+1)\,\delta_n{}^3;\quad n=j+1 \tag{3.4}$$
$$j=2,3;\quad i=1,2;\quad K_4(t,r,z)=\int_0^\infty \eta J_2(\eta t)J_2(\eta r)e^{-\eta|z|}\,d\eta$$
$$2\varphi_2(t)=u_{20}{}^1(t,0)-u_{20}{}^2(t,0),\quad 2\varphi_3(t)=\frac{1}{t^2}\int_0^t t^2\tau(t)\,dt$$
$$\tau(t)=-\frac{\partial}{\partial z}[u_{30}{}^1(t,0)-u_{30}{}^2(t,0)],\quad 0\leqslant t<1;\quad K_3(t,r,z)=$$
$$K_2(t,r,z)$$

where the kernel $K_2(t,r,z)$ is described in relationships (3.1) and $\delta_j{}^i$ is the Kronecker delta.

The conditions

$$\varphi_2(1)=0,\ \varphi_3(1)=0 \tag{3.5}$$

are imposed on the functions $\varphi_j(t)$.

The mechanical meaning of the former is that displacements at the end of the inclusion should be continuous, and that of the latter is that the torque of forces applied to the inclusion is zero.

To determine $\varphi_j(t)\,(j=2,3)$ from (2.12) and (2.14) we obtain the integral equations

$$\varkappa_j\varphi_j(r)+g(r)\int_0^1 t\varphi_j(t)K_j{}^*(t,r)\,dt=f_j{}^*(r),\quad 0\leqslant r<1,\quad j=2,3 \tag{3.6}$$
$$K_j{}^*(t,r)=\int_0^\infty \eta^2 J_k(\eta t)J_k(\eta r)\,d\eta,\quad k=j-1;$$
$$f_2{}^*(r)=g(r)\frac{\partial}{\partial z}u^*(r,0)$$
$$f_3{}^*(r)=g(r)\,r\frac{\partial}{\partial r}\left[\frac{1}{r}u_*(r,0)\right];\quad \varkappa_2=2\gamma_2,\quad \varkappa_3=\frac{2}{\gamma_3}$$

By analogy with /7/ it can be shown that (b is a coefficient in the asymptotic form (1.1))

$$u_j(\rho,\ \varphi)\backsim D_{j0}+\rho^\nu\,[C_{j1}\sin\nu\varphi+D_{j1}\cos\nu\varphi]+\ \ldots, \tag{3.7}$$
$$\rho\to 0,\ 0\leqslant\varphi\leqslant 2\pi$$
$$\nu=\nu_j,\ j=2,\ 3;\ C_{21}=-\mathrm{ctg}\,(\pi\nu_2)\,D_{21},\ D_{31}=\mathrm{ctg}\,(\pi\nu_3)\,C_{31}$$
$$u_j(r,\ z)=u_{j0}{}^1(r,\ z),\ z\geqslant 0;\ u_j(r,\ z)=u_{j0}{}^2(r,\ z),\ z\leqslant 0$$
$$\nu_j\,\mathrm{ctg}\,(\pi\nu_j)+\varkappa_j/b=0;\ {}^1/_2\leqslant\nu_j<1;\ j=2,\ 3 \tag{3.8}$$

We apply the method of collocation to the solution of (3.6). Taking (3.5) and (3.7) into account, we represent $\varphi_j(t)$ as an expansion in Jacobi polynomials $P_n^{(\alpha,\beta)}$ /17/

$$\varphi_j(t) = t^{j-1}(1-t^2)^{\nu} \sum_{n=0}^{N-1} A_n{}^j P_n^{(j-1,\,\nu)}(1-2t^2), \quad \nu = \nu_j \tag{3.9}$$
$$j = 2, 3; \quad |t| < 1$$

Substituting (3.9) into (3.6) and equating the left and right sides of the equations at N collection points x_m $(m = 1, 2, \ldots, N)$, we obtain a system of linear algebraic equations to determine the unknown coefficients A_n^j

$$\sum_{n=0}^{N-1} A_n{}^j L_n{}^j(x_m) = f_j{}^*(x_m), \quad 0 < x_m < 1; \quad m = 1, 2, \ldots, N; \quad j = 2, 3 \tag{3.10}$$

Using results from /17, 19/, we have

$$L_n{}^j(r) = r^{j-1}\varkappa_{jn}{}^* F(n + j + {}^3/_2, -n - \nu - {}^1/_2; j; r^2)$$
$$\varkappa_{jn}{}^* = \frac{\Gamma(n+\nu+1)\,\Gamma(n+j+{}^1/_2)}{n!\,\Gamma(n+\nu+{}^1/_2)}$$

($\nu = \nu_j$, $\Gamma(x)$ is the Gamma function, and $F(a, b; c; x)$ is the Gauss hypergeometric function).

The accuracy with which $\varphi_j(t)$ is calculated is monitored by comparing the results obtained for different values of N in (3.10).

We find from (3.7) and (3.9), to determine the coefficients D_{21} and C_{31}

$$D_{21} = 2^{\nu_2} \sum_{n=0}^{N-1} A_n{}^2 P_n^{(1,\,\nu_2)}(1), \quad C_{31} = 2^{\nu_3} \sum_{n=0}^{N-1} A_n{}^3 P_n^{(2,\,\nu_3)}(1) \tag{3.11}$$

4. Internal asymptotic expansion.

The method of constructing the internal asyptotic expansion that refines the state of stress and strain in the neighbourhood of the end of the inclusion is described in /1, 2/. We will retain this same procedure with the sole difference that in place of an exponential scale in ε for the internal variables /1/, we shall use the following scale (this change does not influence the final result):

$$\rho = \varepsilon\rho_*, \quad 1 - r = \rho\cos\varphi, \quad z = \rho\sin\varphi \tag{4.1}$$

Moreover, the expansion mentioned must be constructed separately in each range (1.6) of variation of the parameter γ.

On the basis of the CAE method /1, 20/, we seek the internal expansion in the range in the form

$$u_\theta(\rho, \varphi) \infty v_0{}^i(\rho_*, \varphi) + \varepsilon v_1{}^i(\rho_*, \varphi) + \ldots, \quad \varepsilon \to 0, \quad i = 0, 1 \tag{4.2}$$

where the functions $v_k{}^0$ are given in the domain W while $v_k{}^1$ are given in the domain Ω where

$$W = \{\rho_* \geqslant 0,\ \alpha_2 \leqslant \varphi \leqslant \alpha_1\}, \quad \Omega = \{\rho_* \geqslant 0,\ \alpha_1 \leqslant \varphi \leqslant 2\pi + \alpha_2\} \tag{4.3}$$
$$R = \Omega \cup W, \quad \alpha_i = \operatorname{arctg}(\varepsilon b_i), \quad \alpha = \alpha_1 - \alpha_2 = \varepsilon b + o(\varepsilon)$$
$$\varepsilon \to 0$$

Rewriting (1.2), (1.3) and (1.4) in internal variables and splitting the differential operators in integer powers of ε, we find that $v_k{}^i$ are the eigensolutions of the Laplace equation in a composite angle R that satisfy the conjugate conditions on the interfacial line of the materials. Using the results from /21/, we find by the CAE method that the eigenfunctions mentioned have the form

$$v_0{}^i(\rho_*, \varphi) = D_0, \quad v_1{}^i(\rho_*, \varphi) = D_{10} + D_1\rho_*^{\lambda_{11}}\cos(\lambda_{11}\varphi) + \gamma^{-\varkappa}C_1\rho_*^{\lambda_{12}}\sin(\lambda_{12}\varphi), \quad \varkappa = \delta_i{}^0; \quad i = 0, 1 \tag{4.4}$$

Here D_{10}, D_i, C_i are coefficients in the asymptotic expansions (3.2) and (3.3), and λ_{11}, λ_{12} are roots close to one for the transcendental equation

$$\cos(\lambda\alpha)\cos[\lambda(2\pi - \alpha)] - {}^1/_2(\gamma + 1/\gamma)\sin(\lambda\alpha)\sin[\lambda(2\pi - \alpha)] = 1 \tag{4.5}$$

whose values we will determine below.

Since the aperture angle of the edge of the inclusion $\alpha \approx \varepsilon b$ is a small quantity it is convenient to seek the solutions of (4.6) in the form of an expansion in the small parameter ε in each of the ranges of variation (1.6) for γ separately. In procedural respects, the expansions mentioned are found exactly as the external representations are determined for the problem under consideration in Sect.2. Therefore, we have

$$\lambda_{11} = 1 + \frac{\varepsilon b}{2\pi}(1 - \gamma) + \ldots, \quad \lambda_{12} = 1 + \frac{\varepsilon b}{2\pi\gamma}(\gamma - 1) + \ldots \tag{4.6}$$

$$\lambda_j = \nu_j + \varepsilon\nu_{j1} + \ldots, \; j = 2, 3$$
$$\nu_{j1} = 2\nu_j b\,[\gamma_j \sin(2\pi\nu_j) - b\nu_j \cos(2\pi\nu_j)]\,[(4\pi\gamma_j - b)\sin(2\pi\nu_j) - b\nu_j\cos(2\pi\nu_j)]^{-1}$$

The subscript j indicates the range of variation of the parameter γ while $\nu_j\,(j = 2, 3)$ are roots of (3.8).

On the basis of (4.2) and (4.4) for the displacements and stresses in the matrix near the edge of the inclusion, we obtain the following approximate expressions written in the variables ρ, φ in the range *1*

$$u_\theta(\rho, \varphi) \approx D_0 + D_1\rho^{\lambda_{11}}\cos(\lambda_{11}\varphi) + C_1\rho^{\lambda_{12}}\sin(\lambda_{12}\varphi), \; \rho \to 0 \tag{4.7}$$
$$\begin{pmatrix}\tau_{\theta z}\\ \tau_{r\theta}\end{pmatrix} \approx \mu\lambda_{11}D_1\rho^{\lambda_{11}-1}\begin{pmatrix}\sin\\ -\cos\end{pmatrix}[(1-\lambda_{11})\varphi] + \mu\lambda_{12}C_1\rho^{\lambda_{12}-1}\begin{pmatrix}\cos\\ \sin\end{pmatrix}[(1-\lambda_{12})\varphi]$$

Relationships to compute the state of stress and strain in the neighbourhood of the edge of the inclusion in ranges *2* and *3* are also obtained by the method described above

$$u_\theta(\rho, \varphi) \approx D_0 + D_{j0} + \rho^\lambda\,[C_{j1}\sin(\lambda\varphi) + D_{j1}\cos(\lambda\varphi)], \; \lambda = \lambda_j \tag{4.8}$$
$$\begin{pmatrix}\tau_{\theta z}\\ \tau_{r\theta}\end{pmatrix} \approx \mu\lambda\rho^{\lambda-1}\left\{C_{j1}\begin{pmatrix}\cos\\ \sin\end{pmatrix}[(1-\lambda)\varphi] + D_{j1}\begin{pmatrix}\sin\\ -\cos\end{pmatrix}[(1-\lambda)\varphi]\right\}, \quad \rho \to 0$$

(the coefficients $C_{j1}, D_{ji}\,(i = 0, 1)$ are defined in (3.7), and j is the subscript indicating the range (1.6) of the parameter γ).

As is well-known /1/, the external and internal asymptotic expansions obtained above overlap with a definite degree of accuracy in a broad domain so that the external expansion can be used, for instance, in the domain $\rho \geqslant \varepsilon$, and the internal expansion for $\rho \leqslant \varepsilon$.

5. Example. Let an elastic system be twisted under the effect of a concentrated moment M applied on the z axis at a height z_0 above the inclusion. In this case /10/

$$u_*(r, z) = \alpha_* r\,[r^2 + (z - z_0)^2]^{-3/2}, \quad \alpha_* = M/(8\pi\mu)$$
$$D_0 = \alpha_*(1 + z_0^2)^{-3/2}, \quad C_1 = 3\alpha_* z_0(1 + z_0^2)^{-5/2}, \quad D_1 = C_1/z_0 - D_0$$

where D_0, C_1, D_1 are the coefficients in (3.3).

The state of stress and strain of a composite in the neighbourhood of the edge of an inclusion is determined in ranges *2* and *3* by relationships (4.8). The coefficients C_{j1}, D_{j1} $(j = 2, 3)$ in these relationships are determined during the solution of external problems (Sect. 3).

The dependences of the coefficients C_{21} and D_{21} on γ are illustrated by curves *1* and *2* in Fig.1 for $\varepsilon = 0.005$ and $\varepsilon = 0.01$ (the dashed and solid lines, respectively); $\alpha_* = 10, z_0 = 2$, $g(r) = 2(1 - r^2)$. The following regularity is seen: as $\gamma_2 = \gamma/\varepsilon \to \infty, \varepsilon \to 0$ the coefficient $C_{21} \to C_1$ (line *3*), while $D_{21} \to 0$. Such a regularity holds in all examples of the axisymmetric torsion of analogous elastic systems and follows from the presence of a range of variation of the parameter γ in which the solutions obtained in cases *1* and *2* overlap with a definite degree of accuracy. Indeed, by taking account of the regularity presented above, as well as by comparing values of the tangential displacements determined by means of (4.7) and (4.8), we see that the difference between these values in the overlap range mentioned above has the form $(1 + C\rho^\lambda)\,O(\varepsilon)$, $\rho \to 0, \varepsilon \to 0$, where $C = \text{const}$, and λ is the root of (4.5).

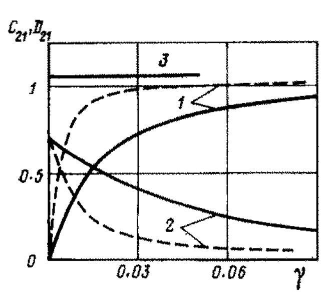

Fig.1

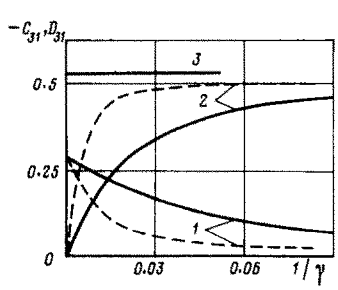

Fig.2

An analogous regularity also holds for the coefficients C_{31}, D_{31}, whose dependence on $1/\gamma$ for the above-mentioned values of $\varepsilon, g(r), \alpha_*$ and z_0 is displated in Fig.2 by curves *1* and *2*, respectively. For $\gamma_3 = \varepsilon\gamma \to \infty, \varepsilon \to 0$ we have $C_{31} \to 0, D_{31} \to D_1$ (line *3*), which indicates the existence of a domain of values of γ, in which the internal asymptotic representations obtained in ranges *1* and *3* agree to within quantities of lower order than the order of the principal terms found for the internal asymptotic expansions.

Note that the internal expansion obtained is not a boundary layer but just refines the index of the stress singularity near the inclusion edge.

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